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LETTER TO THE EDITOR

A nonlinear version of the equivalent bifurcation lemma

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Abstract. We provide an extension of the equivariant bifurcation lemma to the case of nonlinear Lie point symmetries. In this extension, the role of the fixed subspaces under the symmetry subgroups is played by some well specified flow-invariant manifolds. Some typical examples are also considered.

The 'equivariant bifurcation lemma' (see [1–5] and references therein) has become a useful tool in bifurcation problems, and more in general in nonlinear time-evolution problems [6] in the presence of linear symmetries: essentially, it allows one to select some linear subspace, characterized by the property of being the fixed subspace of a symmetry subgroup, and to restrict the original problem to this subspace (in many cases, one or two dimensional).

In this letter, we attempt to extend this result to the case of *nonlinear* Lie point symmetries, i.e. geometrical symmetries (not necessarily linear), as originally introduced by Lie and recently reconsidered by Ovsjannikov, Olver and many others (e.g. [4, 7–9]). We shall show that this extension is actually possible, and very interesting; the main difficulty being that in this case the analogue of the fixed subspace may be given by a more or less singular manifold.

We will consider n-dimensional autonomous bifurcation problems, in the standard form

$$\dot{u} = F(\lambda, u) \qquad 2cmu = u(t) \in R^n \qquad \lambda \in R \tag{1}$$

where $F : \Lambda \times \Omega \to R^n$ is a smooth vector field defined in some neighbourood $\Lambda \times \Omega$ of the origin in $R \times R^n$. As usual in bifurcation theory, we assume there is the trivial stationary solution $u_0 \equiv 0$, i.e.

$$F(\lambda, 0) = 0 \qquad \forall \lambda \in \Lambda \tag{2}$$

and we look for non-trivial stationary or periodic solutions of (1) bifurcating from $u_0 \equiv 0$, as the control parameter λ crosses some critical value $\lambda = \lambda_0$. We can assume $\lambda_0 = 0$.

Let us state the relevant result concerning Lie point symmetries (cf [7, 10]) given by problem (1) in the form most suitable for the applications below. We use the notation (sum over repeated indices, $i, j = 1, \ldots, n$)

$$\partial_t = \partial/\partial t \qquad \partial_i = \partial/\partial u_i \qquad F\partial_u = F_i \partial_i \, .$$

Proposition 1. The symmetries admitted by problem (1) are generated by Lie operators η of the following types (or by linear combinations of them):

(i) $\eta = \Phi(\lambda, t, u)\ (\partial_t + F\partial_u)$ where Φ is an arbitrary function;

(ii) $\eta = \theta_i \partial_i$, where $\theta_i = \theta_i(\lambda, u)$ satisfy the 'determining equations'

$$\theta_j\, \partial_j F_i = F_j\, \partial_j \theta_i \tag{3}$$

(iii) $\eta = \partial_t$;

(iv) if η is any symmetry generator, and $h = h(\lambda, t, u)$ is any integral of motion (depending on time or not) of (1), then also

$$\eta' = \Psi(h)\, \eta$$

where Ψ is arbitrary, generates a symmetry for (1).

Remarks. 1. One could also consider other symmetry generators containing the operator $\partial/\partial\lambda$, i.e. involving changes of the parameter. These are introduced and analysed in [10, 11], but are not relevant to the present discussion.

2. Given any solution $\bar{u} = \bar{u}(t, \lambda, u_0)$, with fixed λ and initial datum u_0, each symmetry of type (i) produces 'motions' $(t, \bar{u}) \to (t + \epsilon, \bar{u} + \epsilon F)$ along $\bar{u}$, leaving invariant (globally, in general, not pointwise) the orbit of $\bar{u}$. They essentially correspond to a 'reparametrization' of time, and are not interesting to our purposes.

3. Putting

$$\widehat{F} \equiv F\partial_u$$

condition (3) for the time-independent Lie point symmetries of type (ii) can be written in the form of a Lie commutator

$$[\widehat{F}, \eta] = 0\,. \tag{4}$$

These symmetries generalize the concept of the 'equivariance' property under linear group representations (cf [10]).

4. Condition (3) (or (4)) is certainly satisfied choosing

$$\eta = \widehat{F} \qquad \text{i.e. } \theta_i \equiv F_i\,. \tag{5}$$

But along any solution $\bar{u}$, one has

$$\widehat{F} \equiv \partial_t \tag{6}$$

the symmetry $\widehat{F}$ then generates the time translations $\bar{u} \to \bar{u}(t + \epsilon)$.

We can finally give the main result of this letter (for an analysis of the algebraic features and some general results based on Lie point symmetries in nonlinear time-evolution problems, see also [10–12]).

Given any Lie point symmetry generator $\eta = \theta_i(\lambda, u)\partial_i$, and any fixed $\lambda \in \Lambda$, let $U_\eta = U_\eta(\lambda)$ be the manifold defined by

$$U_\eta(\lambda) = \{u \mid \theta_i(\lambda, u) = 0,\ i = 1, \ldots, n\}. \tag{7}$$

We then have the following.

Lemma 1. If η_1 is a time-independent Lie point symmetry generator commuting with η, i.e. $[\eta_1, \eta] = 0$, then

$$\eta_1 : U_\eta \to TU_\eta$$

TU_η being the tangent bundle of U_η.

Proof. Let $x \in U_\eta$ and $G_x = \{g \in G \mid gx = x\}$, where G is the symmetry group of (1), i.e. the connected Lie group generated by the Lie point symmetries of (1). If $y = g_1 x$, then $g_1 g_x g_1^{-1} \in G_y$, for each $g_x \in G_x$; in particular if g_1 is a group element generated by η_1, and g_x by η, we have $\theta(\lambda, y) = 0$.

Lemma 2. The manifold $U_\eta(\lambda)$ is invariant, for each fixed λ, under the flow of (1).

Proof. It is sufficient to apply lemma 1 and remarks 3 and 4.

Remark 5. As a consequence of (iv) in the above proposition, if h is a time-independent integral of motion, and if $\eta' = \Psi(h)\eta$, condition (7) can be met through the condition $\Psi(h) = 0$, i.e. $h =$ constant. In this case, the result that the manifold $U_{\eta'}$ is flow invariant is, in fact, obvious. However, some examples below will show some non-trivial aspects of this situation.

Clearly, the above lemmas allow the reduction of the original problem, if the initial datum is in U_η, to a problem lying in the lower-dimensional manifold U_η. Now, under reasonable hypotheses on these manifolds, and on the restricted problem, many of the typical results of bifurcation theory can be easily recovered. For instance, we can say:

Let $U = U_\eta(\lambda)$, a manifold satisfying (7), be one dimensional (for each fixed $\lambda \in \Lambda$), and assume that $u_0 \equiv 0 \in U$ for any $\lambda \in \Lambda$. Assume there is a smooth parametrization of $U_\eta(\lambda)$ for each λ such that $U_\eta(\lambda)$ is diffeomorphic to R:

$$u_i = u_i(\lambda, s) \qquad s \in R \qquad u_i(\lambda, 0) = 0$$

then the restriction of the right-hand side of (1) to U becomes

$$F(\lambda, u(\lambda, s)) \equiv \mathcal{F}(\lambda, s) = 0 \tag{8}$$

and the original problem becomes just a standard one-dimensional bifurcation problem, with, in particular, $\mathcal{F}(\lambda, 0) = 0$.

Similarly, if $U = U_\eta(\lambda)$ is two dimensional, for each λ, if $0 \in U$, and U is diffeomorphic to R^2, then usual hypotheses (e.g. the standard Hopf hypothesis [3, 5, 13]) can ensure the existence of a periodic bifurcation on U. Different possibilities, for both the one- and two-dimensional cases, are considered in [12]. The assumptions about the regularity of the manifolds U can be substituted by different assumptions, concerning e.g. stability properties [14–16] or by arguments *à la* Poincaré–Bendixson [14,17]. For instance, if one assumes that for $\lambda = \lambda_0 = 0$ the solution $u_0 \equiv 0$ on the manifold is asymptotically stable, whereas for $\lambda > 0$ it becomes completely unstable, then a stable bifurcation appears for $\lambda > 0$; if U is two dimensional, the bifurcation sets are limit cycles, corresponding to either a periodic solution or a set of stationary points.

We propose now a list of examples, that—even if very simple—may give an idea of some of the different situations covered by the above discussion.

Example 1. This is taken from standard *linear* theory, and is given here to clarify how the results of the linear group representation theory can be rephrased in terms of the above language. Let the group G = SO(3) operate on the five-dimensional space of real symmetric traceless 3 × 3 matrices M according to the linear representation

$$M \to M' = gMg^{-1} \qquad g \in \mathrm{SO}(3). \tag{9}$$

Putting $M = u_i e_i$, $i = 1, \ldots, 5$, where

$$e_1 = 2^{-1/2}\ \mathrm{diag}\ (1,-1,0) \qquad e_2 = \mathrm{diag}\ (1,1,-2)$$

and e_3, e_4, e_5 span the subspace of the off-diagonal matrices, it is easy to see that the Lie generator η_3 of the rotations around the third axis has in this space the following differential form

$$\eta_3 = u_3\partial_1 - u_1\partial_3 + u_5\partial_4 - u_4\partial_5\,.$$

Then, condition (7) gives that the space $U_\eta \equiv \{\alpha e_2, \alpha \in R\}$ is invariant under the flow of any problem equivariant under the group action (9); this is precisely the result obtained from linear theory: U_η is in fact the fixed subspace under the isotropy subgroup generated by η_3.

Example 2. Let now $u \equiv (x,y) \in R^2$, and consider the problem

$$\dot{x} = f(\lambda, x)$$
$$\dot{y} = 2\lambda x f(\lambda, x) - g(\lambda, x)\,(y - \lambda x^2)$$

where f, g are arbitrary functions. The problem admits the symmetry generated by

$$\eta = (y - \lambda x^2)\,\partial_y \tag{10}$$

and for each fixed λ there is the flow-invariant manifold $U = \{(x,y) \mid y = \lambda x^2\}$. If e.g. $f = \lambda x - x^2$, there is a stationary bifurcation given by

$$x = s \qquad y = s^3 \qquad \lambda = s \qquad (s \in R)\,.$$

Example 3. This and the following examples may illustrate the role played by integrals of motion in this context. With $u \equiv (x,y) \in R^2$, and $r^2 = x^2 + y^2$, consider

$$\dot{x} = \lambda x - xr^2 - \epsilon\, y$$
$$\dot{y} = \lambda y - yr^2 + \epsilon\, x.$$

Both for $\epsilon = 0$ and $\epsilon \neq 0$, this problem possesses the obvious rotation symmetry

$$\eta = x\partial_y - y\partial_x\,. \tag{11}$$

Applying condition (7) to this operator, we obtain only the origin $(0,0)$. According to remark 5, we look now for (time-independent) integrals of motion. If $\epsilon = 0$, we obtain $y/x =$ constant, and, in fact, in each straight line from the origin there is stationary bifurcation. If $\epsilon \neq 0$, we have instead

$$\frac{\lambda - r^2}{r^2} \exp\left(\frac{2\lambda}{\epsilon} \tan^{-1}\frac{y}{x}\right) = \text{constant}$$

which defines (for each fixed $\lambda \neq 0$) a very singular one-dimensional manifold. Notice in particular the singularity for $x, y \to 0$, and, actually, no stationary bifurcation exists along these manifolds (clearly, there is instead a Hopf bifurcation, and the existence of limit cycles shows that this singular behaviour cannot be avoided, according to well known arguments [14, ch 11, section 5]). It can also be noted that the two cases $\epsilon = 0$ and $\epsilon \neq 0$ actually possess different symmetry groups (O(2) and SO(2) respectively), which cannot be distinguished at the algebraic level (equation (11)), but are distinguished by the different form of the respective flow-invariant manifolds.

Example 4. Let $u \equiv (x, y, z) \in R^3$, and consider problems of the following form

$$\dot{x} = xf(\lambda, z) - yg(\lambda, z)$$
$$\dot{y} = yf(\lambda, z) + xg(\lambda, z)$$
$$\dot{z} = zf(\lambda, z).$$

Symmetries for these problems are clearly generated by

$$\eta_1 = x\partial_y - y\partial_x \qquad \eta_2 = x\partial_x + y\partial_y\,. \tag{12}$$

Condition (7) applied to either η_1 or η_2 gives for U_η just the z axis, thus obtaining a standard one-dimensional problem. Time-independent integrals of motion of the above problem assume constant values along the cones

$$x^2 + y^2 = \text{constant} \times z^2 \tag{13}$$

and respectively the surfaces

$$\tan^{-1}(y/x) + \chi(\lambda, z) = \text{constant} \tag{14}$$

where χ is a function depending on f, g. Despite the singularities, there may be bifurcation (depending on the functions f, g). For example, if for $z = z_0$, there is $\lambda = \lambda(z_0)$ such that $f(\lambda(z_0), z_0) = 0$, with $\lambda(z_0) \to 0$ when $z_0 \to 0$, then each cone (13) contains a bifurcating solution at the level $z = z_0$. If $g(\lambda(z_0), z_0) \neq 0$, this is a periodic solution (with period $2\pi/g(\lambda(z_0), z_0)$); if $g(\lambda(z_0), z_0) = 0$, this is a set of stationary solutions. The other manifold (14) does not contain, in general, periodic bifurcating solutions.

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